

On “*Exponential Lower Bounds for Polytopes in Combinatorial Optimization*” by Fiorini *et al.* (2015)*: A Refutation For Models With Disjoint Sets of Descriptive Variables

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Abstract. We provide a numerical refutation of the developments of Fiorini *et al.* (2015)* for models with disjoint sets of descriptive variables. We also provide an insight into the meaning of the existence of a one-to-one linear map between solutions of such models.

Keywords: Linear Programming; Combinatorial Optimization; Traveling Salesman Problem; TSP; Computational Complexity.

*: Fiorini, S., S. Massar, S. Pokutta, H.R. Tiwary, and R. de Wolf (2015). Exponential Lower Bounds for Polytopes in Combinatorial Optimization. *Journal of the ACM* 62:2, Article No. 17.

1 Introduction

Extended formulations (EFs) have been the dominant theory which has been used in deciding on the validity of proposed LP models for hard combinatorial problems. All of the developments are predicated on the model being evaluated *projecting* to the “natural” polytope of the specific problem at hand. We show in Diaby and Karwan (2016a and 2016b, respectively) that the developments of Fiorini *et al.* (2011; 2012) are not applicable in relating polytopes which are described in terms of disjoint sets of variables. In the journal version of their papers, Fiorini *et al.* (Fiorini *et al.* (2015)) have added the stipulation that the polytopes they consider be of dimensions greater than zero (e.g., see p. 17:9, last sentence of “Lemma 2”; p. 17:17, first sentence of “Theorem 13”). The objective of this technical note is to show that the counter-examples provided in Diaby and Karwan (2016a and 2016b, respectively) for the Fiorini *et al.* (2011; 2012) developments remain valid for the Fiorini *et al.* (2015) work. We do this by providing a counter-example using polytopes of dimensions greater than zero which similarly refute the Fiorini *et al.* (2015) developments. For convenience, we start with a recall of the standard definition of an “*extended formulation*” as well as those of the alternate definitions used in Fiorini *et al.* (2015). Then, we discuss our numerical example. Finally, we provide some insight into the meaning of the existence of a one-to-one correspondence between solutions of models when the models have disjoint sets of descriptive variables.

2 Background Definitions

Definition 1 (“Standard EF Definition”) An *extended formulation* for a polytope $X \subseteq \mathbb{R}^p$ is a polyhedron $U = \left\{ \begin{pmatrix} w \\ x \end{pmatrix} \in \mathbb{R}^{p+q} : Gx + Hw \leq g \right\}$ the projection, $\varphi_x(U) := \{x \in \mathbb{R}^p : (\exists w \in \mathbb{R}^q : \begin{pmatrix} w \\ x \end{pmatrix} \in U)\}$, of which onto x -space is equal to X (where $G \in \mathbb{R}^{m \times p}$, $H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$) (Yannakakis (1991)).

Definition 2 (“Fiorini *et al.* Definition #1”) A polyhedron $U = \left\{ \begin{pmatrix} w \\ x \end{pmatrix} \in \mathbb{R}^{p+q} : Gx + Hw \leq g \right\}$ is an *extended formulation* of a polytope $X \subseteq \mathbb{R}^p$ if there exists a linear map $\pi : \mathbb{R}^{p+q} \rightarrow \mathbb{R}^p$ such that X is the image of U under π (i.e., $X = \pi(U)$; where $G \in \mathbb{R}^{m \times p}$, $H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$) (see Fiorini *et al.* (2015; p. 17:3, lines 20-21; p. 17:9, lines 22-23)).

Definition 3 (“Fiorini *et al.* Definition #2”) An *extended formulation* of a polytope $X \subseteq \mathbb{R}^p$ is a linear system $U = \left\{ \begin{pmatrix} w \\ x \end{pmatrix} \in \mathbb{R}^{p+q} : Gx + Hw \leq g \right\}$ such that $x \in X$ if and only if there exists $w \in \mathbb{R}^q$ such that $\begin{pmatrix} w \\ x \end{pmatrix} \in U$. (In other words, U is an EF of X if $\left(x \in X \iff \left(\exists w \in \mathbb{R}^q : \begin{pmatrix} w \\ x \end{pmatrix} \in U \right) \right)$ (where $G \in \mathbb{R}^{m \times p}$, $H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$) (see Fiorini *et al.* (2015; p. 17:2, last paragraph; p. 17:9, line 20-21)).

3 Numerical refutation of Fiorini *et al.* (2015)

Our numerical counter-example will now be discussed.

Example 4 : Let $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}$ be disjoint vectors of variables. Let X be a polytope in the space of \mathbf{x} , and U , a polytope in the space of $\begin{pmatrix} \mathbf{w} \\ \mathbf{x} \end{pmatrix}$, with:

$$X := \text{Conv} \left(\left\{ \begin{pmatrix} 8 \\ 10 \\ 6 \end{pmatrix}, \begin{pmatrix} 12 \\ 15 \\ 9 \end{pmatrix} \right\} \right), \text{ and} \quad (1)$$

$$U := \left\{ \begin{pmatrix} \mathbf{w} \\ \mathbf{x} \end{pmatrix} \in \mathbb{R}^4 : 2 \leq \mathbf{0} \cdot \mathbf{x} + \mathbf{w} \leq 3 \right\} \quad (2)$$

We now discuss some key results of Fiorini *et al.* (2015) which are refuted by X and U .

1. Refutation of the validity of Definition 2.

(a) Note that the following is true for X and U :

$$\left(\mathbf{x} \in X \nRightarrow \left(\exists \mathbf{w} \in \mathbb{R} : \begin{pmatrix} \mathbf{w} \\ \mathbf{x} \end{pmatrix} \in U \right) \right). \quad (3)$$

For example,

$$\left(\exists \mathbf{w} \in \mathbb{R} : \begin{pmatrix} w \\ 22.5 \\ -50 \\ 100 \end{pmatrix} \in U \right) \not\Rightarrow \left(\begin{pmatrix} 22.5 \\ -50 \\ 100 \end{pmatrix} \in X \right). \quad (4)$$

Hence, U is **not** an *extended formulation* of X according to Definition 3.

(b) Observe that the following is also true for X and U :

$$X = \left\{ \mathbf{x} \in \mathbb{R}^3 : \left(\mathbf{x} = A \cdot \begin{pmatrix} \mathbf{w} \\ \mathbf{x} \end{pmatrix}, \begin{pmatrix} \mathbf{w} \\ \mathbf{x} \end{pmatrix} \in U \right) \right\}, \text{ where } A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}. \quad (5)$$

In other words, X is the image of U under the linear map A . Hence, U is an *extended formulation* of X according to Definition 2.

(c) It follows from (a) and (b) above, that Definitions 2 and 3 are in contradiction of each other with respect to X and U . Hence X and U are a refutation of the validity of Definition 2 (since it is easy to verify the equivalence of Definition 3 to Definition 1, which is the “standard” definition).

2. Refutation of “Theorem 3” (p.17:10) of Fiorini et al. (2015).

The proof of the theorem (“Theorem 3”) hinges on Definition 3. The specific statement in Fiorini et al. (2015; p. 17:10, lines 26-28) is:

“...Because

$$Ax \leq b \iff \exists y : E^=x + F^=y = g^=, E^{\leq}x + F^{\leq}y = g^=, \quad (6)$$

each inequality in $Ax \leq b$ is valid for all points of Q”

The equivalent of (6) in terms of X and U is:

$$\mathbf{x} \in X \iff \exists \mathbf{w} \in \mathbb{R} : \begin{pmatrix} \mathbf{w} \\ \mathbf{x} \end{pmatrix} \in U. \quad (7)$$

Clearly, (7) is **not true**, as we have illustrated in Part (1.a) above. Hence, the proof of “Theorem 3” (and therefore, “Theorem 3”) of Fiorini et al. (2015) is refuted by X and U .

3. Refutation of “Lemma 9” (p. 17:13-17:14) of Fiorini et al. (2015).

The first part of the lemma is stated (in Fiorini et al. (2015)) thus:

“Lemma 9. Let P , Q , and F be polytopes. Then, the following hold:

(i) if F is an extension of P , then $\text{xc}(F) \geq \text{xc}(P)$;...”

The proof of this is stated as follows:

“Proof. The first part is obvious because every extension of F is in particular an extension of P”

The notation “ $xc(\cdot)$ ” stands for “*extension complexity* of (\cdot) ,” which is defined as (p. 17:9, lines 24-25 of Fiorini *et al.* (2015)):

“...the *extension complexity* of P is the minimum size (i.e., the number of inequalities) of an EF of P .”

The refutation of these for X (as shown in (1) above) and U (as shown in (2) above) is as follows.

As shown in Part (1) above, U is an *extension* of X according to Definition 2 (which is central in Fiorini *et al.* (2015)). This means that U is an *extended formulation* of every one of the infinitely-many possible \mathcal{H} -descriptions of X . This would be true in particular for the \mathcal{H} -description below for X :

$$X := \left\{ \begin{array}{l} \mathbf{x} \in \mathbb{R}^3 : \\ -5x_1 + 4x_2 \leq 0; \\ 3x_2 - 5x_3 = 0; \\ 3x_1 - 4x_3 \leq 0; \\ 8 \leq x_1 \leq 12; \\ 10 \leq x_2 \leq 15; \\ 6 \leq x_3 \leq 9 \end{array} \right\}. \quad (8)$$

Clearly, however, we have that:

$$xc(U) \not\leq xc(X). \quad (9)$$

Hence, X and U are a refutation of “Lemma 9” of Fiorini *et al.* (2015), being that U is the *extension*, and X , the *projection*, according to definitions used in Fiorini *et al.* (2015).

□

According to Fiorini *et al.* (2015; p. 17:7, Section 1.4, first sentence; p. 17:11, lines 6-11; p. 17:14, lines 5-6; p.17:16, lines 13-14 after the “Fig. 4” caption), their “Theorem 3” and “Lemma 9” play pivotal, foundational roles in the rest of their developments. Note that “Lemma 9” (of Fiorini *et al.* (2015)) does not depend on any one of the *extended formulations* definitions used in Fiorini *et al.* (2015) in particular. Hence, we believe the numerical illustration we have provided above represents a simple-yet-complete refutation of their developments when polytopes are described in terms of disjoint sets of variables. In other words, our counter-example shows that the Fiorini *et al.* (2015) developments may be valid for models which require (in a non-redundant way) the natural variables used to describe the standard polytopes of the problems they consider only.

As can be seen from the illustrative counter-example above, the existence of a linear map stipulated in Definition 2 is not sufficient to imply that there exists an *extended formulations* relationship between the models from which valid/meaningful inferences can be made. We will provide, below, some insights into the correct meaning/consequence of the existence of a linear map stipulated in Definition 2, establishing a one-to-one correspondence between two models that are stated in disjoint variable spaces.

4 Meaning of the existence of a linear transformation

We will focus on the more general case (than that of the linear map stipulated in Definition 2) of the existence of an affine transformation which was brought to our attention in private e-mails by Yannakakis (2013). In the case of polytopes stated in disjoint variable spaces, if the constraints expressing the affine transformation are *redundant* for each of the models/polytopes, the implication is that one model can be used in an “auxiliary” way, in order to solve the optimization problem over the other model, without any reference to/knowledge of the \mathcal{H} -description of that other model. This is shown in Remark 5 below.

Remark 5

- Let:

- $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ be disjoint vectors of variables;
- $X := \{x \in \mathbb{R}^p : Ax \leq a\}$;
- $L := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{p+q} : Bx + Cy = b \right\}$;
- $Y := \{y \in \mathbb{R}^q : Dy \leq d\}$;

(Where: $A \in \mathbb{R}^{k \times p}$; $a \in \mathbb{R}^k$; $B \in \mathbb{R}^{m \times p}$; $C \in \mathbb{R}^{m \times q}$; $b \in \mathbb{R}^m$; $D \in \mathbb{R}^{l \times q}$, $d \in \mathbb{R}^l$).

- If $B^T B$ is nonsingular, then L can be re-written in the form:

$$L = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{p+q} : x = \bar{C}y + \bar{b} \right\}.$$

$$(\text{Where: } \bar{C} := -(B^T B)^{-1} B^T C, \text{ and } \bar{b} := (B^T B)^{-1} B^T b). \quad (10)$$

Hence, the linear map stipulated in Definition 2 is simply a special case of L in which $b = 0$ and $B^T B$ is nonsingular.

- Assume that:

- $L \neq \emptyset$ exists, with constraints that are *redundant* for X and Y , respectively;
- the non-negativity requirements for x and y are included in the constraint sets of X and Y , respectively; and that:
- $B^T B$ is nonsingular.

(This is equivalent to assuming that the more general (affine map) version of the linear map stipulated in Definition 2 exists.)

- Then, the optimization problem:

Problem LP₀:

$$\left| \begin{array}{ll} \text{Minimize:} & \alpha^T x \\ \text{Subject To:} & x \in X \\ \text{(where } \alpha \in \mathbb{R}^p \text{).} \end{array} \right.$$

is equivalent to:

Problem LP₁:

$$\left| \begin{array}{ll} \text{Minimize:} & \alpha^T x \\ \text{Subject To:} & \begin{pmatrix} x \\ y \end{pmatrix} \in L; \ x \in X; \ y \in Y \\ & (\text{where } \alpha \in \mathbb{R}^p). \end{array} \right.$$

which is equivalent to the smaller problem:

Problem LP₂:

$$\left| \begin{array}{ll} \text{Minimize:} & (\alpha^T \overline{C}) y + \alpha^T \overline{b} \\ \text{Subject To:} & y \in Y \\ & (\text{where } \alpha \in \mathbb{R}^p). \end{array} \right.$$

- Hence, if L is the graph of a one-to-one correspondence between the points of X and the points of Y (see Beachy and Blair (2006, pp. 47-59)), then, the optimization of any linear function of x over X can be done by first using *Problem LP₂* in order to get an optimal y , and then using Graph L to “retrieve” the corresponding x . Note that the second term of the objective function of *Problem LP₂* can be ignored in the optimization process of *Problem LP₂*, since that term is a constant.

Hence, if L is derived from knowledge of the \mathcal{V} -representation of X only, then this would mean that the \mathcal{H} -representation of X is not involved in the “two-step” solution process (of using *Problem LP₂* and then Graph L), but rather, that only the \mathcal{V} -representation of X is involved.

□

5 Conclusions

We have shown the non-validities of an alternate definition of *extended formulations* (Definition 2) which is used in the Fiorini *et al.* (2011; 2012; 2015) developments when the models being studied are (or can be) stated in terms of disjoint sets of variables, and of key results of theirs (specifically, “Theorem 3” and “Lemma 9” in Fiorini *et al.* (2015)) which are the foundations for all their other developments. Hence, the claims in Fiorini *et al.* (2015) that:

“We solve this question by proving a super-polynomial bound on the number of inequalities in every LP for the TSP.” (Fiorini *et al.* (2015, p.17:2, lines 9-10.));

“We also prove such unconditional super-polynomial bounds for the maximum cut and the maximum stable set problems.” (Fiorini *et al.* (2015, p.17:2, lines 10-12.));

and

“...it is impossible to prove $P = NP$ by means of a polynomial-sized LP that expresses any of these problems.” (Fiorini *et al.* (2015, p.17:2, lines 12-13.))

are not supported by the developments in Fiorini *et al.* (2015), since, as we have shown in this short note, those developments are not valid for (and are, therefore, not applicable to) models that do not need the natural/standard variables that are used to describe the standard polytopes for the problems considered in Fiorini *et al.* (2015). Hence, these claims are overscoped/overreaching, and refuted by the developments in this short note.

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